Name: Rana Salkhnini ; ID 62000

Exam II, MTH 205, Fall 2014

Ayman Badawi

$$\frac{100}{100}$$

QUESTION 1. (10 points) Solve $\frac{dy}{dx} = \frac{1}{-x+2y+y^2}$

$$\cancel{dx(1) - dy(-x+2y+y^2) = 0}$$

$$\frac{dx}{dy} = -x + 2y + y^2$$

$$x' + x = (2y+y^2)$$

$$e^{\int 1 dy} = e^{y}$$

$$x = \frac{\int e^{y}(2y+y^2)dy}{e^{y}} = \frac{e^{y}y^2 + c}{e^{y}}$$

$$x = y^2 + ce^{-y}$$

✓

QUESTION 2. (10 points) Solve $\frac{dy}{dx} = \frac{-9x+3y+e^{(-3x+y)}}{-3x+y}$

$$\frac{dy}{dx} = \frac{3(-3x+y) + e^{(-3x+y)}}{-3x+y}$$

$$w = -3x + y$$

$$\frac{dw}{dx} = -3 + \frac{dy}{dx}$$

$$\frac{dw}{dx} + 3 = \frac{dy}{dx}$$

$$\cancel{3} + \frac{dw}{dx} = \frac{\cancel{3w}}{\cancel{w}} + \frac{e^{w}}{w}$$

$$\frac{dw}{dx} = \frac{e^{w}}{w}$$

$$\int \frac{dw(w)}{e^{w}} = \int dx$$

$$w \quad + \quad e^{-w}$$
$$1 \quad - \quad -e^{-w}$$
$$0 \quad e^{-w}$$

$$c + x = -we^{-w} - e^{-w}$$

$$c + x = -(-3x+y)e^{-(-3x+y)} - e^{-(-3x+y)}$$

✓

Bana

QUESTION 3. (15 points) Solve $y^{(2)} - \frac{10}{x}y' + \frac{18}{x^2}y = \frac{4}{x^{14}}$

yh

$$y = x^n$$
$$y' = n x^{n-1}$$
$$y'' = (n^2-n) x^{n-2}$$
$$(n^2-n)\cancel{x^{n-2}} - 10n\cancel{x^{n-2}} + 18\cancel{x^{n-2}} = 0$$
$$n^2 - n - 10n + 18 = 0$$
$$n = 9 \qquad y_h = c_1 x^2 + c_2 x^9$$
$$n = 2$$

yp

$y_1 = x^2$	$y_1' = 2x$
$y_2 = x^9$	$y_2' = 9x^8$

$$2x\,u'(x) + 9 \cdot x^8 v'(x) = \frac{4}{x^{14}}$$
$$x^2 u'(x) + x^9 v'(x) = 0$$

$$u'(x) = \frac{\begin{vmatrix} \frac{4}{x^{14}} & 9x^8 \\ 0 & x^9 \end{vmatrix}}{\begin{vmatrix} 2x & 9x^8 \\ x^2 & x^9 \end{vmatrix}} = \frac{\frac{4}{x^{14}}(x^9)}{2x^{10} - 9x^{10}} = \frac{\frac{4}{x^5}}{-7x^{10}}$$

$$u'(x) = \frac{4}{-7x^{15}}$$

$$v'(x) = \frac{\begin{vmatrix} 2x & \frac{4}{x^{14}} \\ x^2 & 0 \end{vmatrix}}{\begin{vmatrix} 2x & 9x^8 \\ x^2 & x^9 \end{vmatrix}} = \frac{-\frac{4}{x^{14}}x^2}{-7x^{10}} = +\frac{4}{7x^{22}}$$

$$u(x) = \int \frac{4}{-7x^{15}}\,dx = \frac{4}{+7(14)x^{14}} = \frac{2}{49x^{14}}$$

$$v(x) = \int \frac{4}{7x^{22}}\,dx = -\frac{4}{7(21)x^{21}} = -\frac{4}{147x^{21}}$$

$$Y_g = Y_p + Y_h = c_1x^2 + c_2x^9 + \frac{2}{49x^{14}}(x^2) - \frac{4x^9}{147}\left(\frac{1}{x^{21}}\right)$$

$$c_1x^2 + c_2x^9 + \frac{2}{49x^{12}} - \frac{4}{147x^{12}} = \boxed{c_1x^2 + c_2x^9 + \frac{2}{147x^{12}}}$$

QUESTION 4. (10 points) Solve $y^{(2)} + \frac{-2}{2x+1}y' = \frac{-(2x+1)^2}{(x^2+x+1)^2}$

$\Rightarrow \int \frac{-2}{u} \frac{du}{2}$

$u = 2x+1$

$du = 2\cancel{x}\,dx$

$\frac{du}{2\cancel{x}} = dx$

$w = y'$

$$w' + \frac{-2}{2x+1} w = \frac{-(2x+1)^2}{(x^2+x+1)^2}$$

$$e^{\int \frac{-2}{2x+1}dx} = e^{\int \frac{-1}{u}dx} = e^{-\ln(2x+1)} = \frac{1}{2x+1}$$

$$w = \frac{\int \cancel{\frac{1}{2x+1}}\left(\frac{-(2x+1)^{\cancel{2}}}{(x^2+x+1)^2}\right)dx}{\frac{1}{2x+1}}$$

$$= \frac{-\int \frac{2x+1}{(x^2+x+1)^2}dx}{\frac{1}{2x+1}} = \frac{-\int \frac{1}{u^2}du}{\frac{1}{2x+1}}$$

$u = x^2+x+1$

$du = 2x+1\,dx$

$dx = \frac{du}{2x+1}$

$$= \frac{\frac{1}{x^2+x+1} + c_1}{\frac{1}{2x+1}}$$

$$w = \frac{2x+1}{x^2+x+1} + c_1(2x+1)$$

$$y = \int \frac{2x+1}{x^2+x+1}dx + c_1\int(2x+1)$$

$$= \ln(x^2+x+1) + c_1(x^2+x) + c$$

✓

QUESTION 5. (10 points) Solve $y' + \frac{-1}{3}y = (\frac{-1}{3}cos(3x) + sin(3x))y^4$

$n = 4$

$1 - n = 1 - 4 = -3$

$w = y^{-3}$ $\qquad w = \frac{1}{y^3} \Rightarrow \frac{1}{w} = y^3$

$y = \sqrt[3]{\frac{1}{w}}$

$w' + w = \cos 3x - 3\sin 3x$

$e^{\int 1 dx} = e^x$

$w = \frac{\int e^x (\cos 3x - 3\sin 3x) dx}{e^x} = \frac{e^x \cos 3x + c}{e^x}$

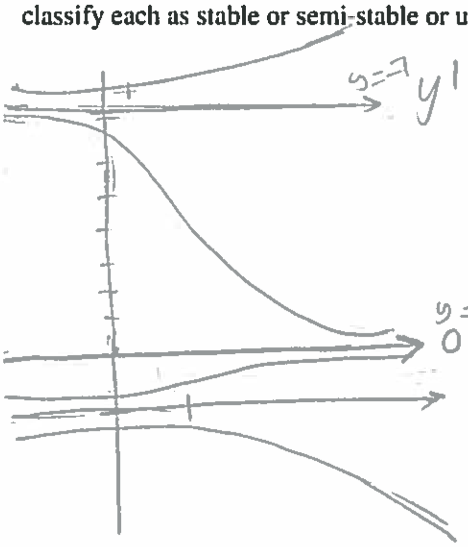
$w = \cos 3x + ce^{-x}$

$y = \sqrt[3]{\frac{1}{\cos 3x + ce^{-x}}}$

✓

Bana

QUESTION 6. (10 points) Given Given $y' = y^3 - 6y^2 - 7y$. Find the critical points. Then classify each as stable or semi-stable or unstable. Roughly, sketch the solution to the DE if $y(0) = -0.5$

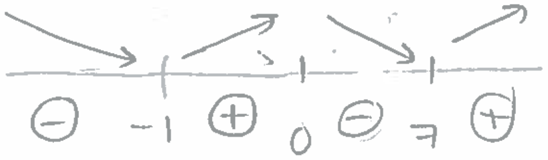


$y' = y(y^2 - 6y - 7) = 0$

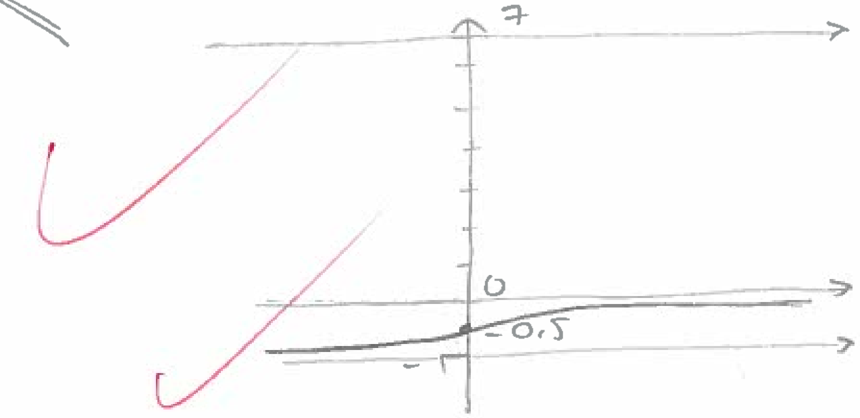
$y = 7$ $y = 0$ $y = -1$ ← unstable

unstable

stable



$y(0) = -0.5$



QUESTION 7. (10 points) Is $y' = \frac{-(y^2+4x^3+e^x+10)}{2yx+e^y+2y-10}$ exact? If yes, then solve it. If no, then find a method that will help us to solve it.

$$\frac{dy}{dx} = \frac{-(y^2+4x^3+e^x+10)}{2yx+e^y+2y-10}$$

$$dy(2yx+e^y+2y-10) + dx(-y^2+4x^3+e^x+10) = 0$$

$2y \overset{?}{=} 2y$ it is exact

$$\frac{dF}{dx} = y^2+4x^3+e^x+10$$

$$F = \int(y^2+4x^3+e^x+10)dx = xy^2+x^4+e^x+10x+h(y)$$

$$\frac{dF}{dy} = \cancel{2xy} + h'(y) = \cancel{2y}x + e^y + 2y - 10$$

$$h'(y) = e^y+2y-10$$

$$h(y) = \int(e^y+2y-10)dy = e^y+y^2-10y+c$$

$$xy^2+x^4+e^x+10x+e^y+y^2-10y+c=0$$

QUESTION 8. (10 points) An ice-cream cake with initial temperature 0C is placed in a room that has constant temperature 20C. If after 2 minutes, the temperature of the cake is 4C. a) How long will it take for the cake to reach the room temperature? b) What is the temperature of the cake after 30 minutes?

$T(0) = 0^\circ C \quad T(2) = 4$

$T_c = 20^\circ C$

$$\frac{dT}{dt} = -k(-T+20)$$

$$\int \frac{dT}{(20-T)} = -\int k\,dt$$

$$\ln(20-T) = -kt + c \leftarrow 2.995$$

$$20e^{-0.111t} = 20-T$$

$$T = 20 - 20e^{-0.112t}$$

a)

$$20 = 20 - 20e^{-0.112t}$$

$$0 \neq -20e^{-0.111t}$$

as $t \to \infty$, it will get close to 20° but never reach it

b)

$$T = 20 - 20e^{-0.112(30)}$$

$$= 19.3^\circ C$$

✓

QUESTION 9. (15 points) Let $A(t)$ be the amount of salt at any time t. A 50-gal tank initially holds 10 gallons of fresh water (i.e. $A(0) = 0$). A mixture containing 1 kg of salt per gallon is poured into the tank at the rate 4 gal/min, while the well stirred mixture leaves the tank at rate 2 gal/min. a) Find A(t). b) Find the amount of salt at the moment of overflow? Find the concentration of salt per gallon after 10 minutes?

10 $\quad A(0)=0$

in 4gal/min. 1kg/g

out 2gal/min

a)

$$\frac{dA}{dt} = 4 - 2\left(\frac{A(t)}{10+4t-2t}\right) = 4 - \frac{1}{5+t}A(t)$$

$$A'(t) + \frac{1}{5+t}A(t) = 4$$

$$e^{\int \frac{1}{5+t}dt} = 5+t$$

$$A(t) = \frac{\int (5+t)4}{5+t} = \frac{4\left(5t+\frac{t^2}{2}\right)+\cancel{c}}{5+t}$$

$$0 = \frac{4(0)+c}{5+0} = \frac{c}{5} \Rightarrow c=0$$

b) $10 + 2t = 50$

$t = 20$

$$A(20) = \frac{4\left(5(20)+\frac{(20)^2}{2}\right)}{5+20} = 48\text{ kg}$$

✓

→ At the back

Name Sneha Sasi, ID 54104

Exam II, MTH 205, Fall 2014

Ayman Badawi

$\frac{100}{100}$

QUESTION 1. (10 points) Solve $\frac{dy}{dx} = \frac{1}{-x+2y+y^2}$

$\frac{dx}{dy} = -x+2y+y^2$

$x' = -x+2y+y^2$

$x'+x = (2y+y^2)$

$e^{\int 1 dy} = e^y$

$x = \frac{\int e^y \cdot (2y+y^2)\, dy}{e^y}$

$= \frac{e^y y^2 + C}{e^y} = y^2 + Ce^{-y}$ ✓

$y^2 = u$

$2y\,dy = du$

$\boxed{x = y^2 + ce^{-y}}$

QUESTION 2. (10 points) Solve $\frac{dy}{dx} = \frac{-9x+3y+e^{(-3x+y)}}{-3x+y}$ $= \frac{+3(-3x+y) + e^{(-3x+y)}}{-3x+y}$

$-3x+y = w$

$-3 + \frac{dy}{dx} = \frac{dw}{dx}$

$\frac{dy}{dx} = \frac{dw}{dx} + 3$

$\frac{dw}{dx} + 3 = \frac{3(w) + e^w}{w}$

$\frac{dw}{dx} + \cancel{3} = \cancel{3} + \frac{e^w}{w}$

$\frac{dw}{dx} = \frac{e^w}{w}$

$\int we^{-w} dw = \int dx$

d	I
w	$+e^{-w}$
1	$-e^{-w}$
0	e^{-w}

$w(-e^{-w}) - e^{-w} = x + C$

$(-3x+y)(-e^{-(-3x+y)}) - e^{-(-3x+y)} = x + C$ ✓

$(-3x+y)(-e^{-(-3x+y)}) - e^{-(-3x+y)} = x + C$

QUESTION 3. (15 points) Solve $y^{(2)} - \frac{10}{x}y' + \frac{18}{x^2}y = \frac{4}{x^{14}}$

$y = x^n$

$y' = n x^{n-1}$

$y'' = n(n-1)x^{n-2}$

for homogenous part:

$(n^2-n)x^{n-2} - \frac{10}{x} n(x^{n-1}) + \frac{18}{x^2} x^n = 0$

$(n^2-n)x^{n-2} - 10n(x^{n-2}) + 18x^{n-2} = 0.$

$n^2 - n - 10n + 18 = 0$

$n^2 - 11n + 18 = 0$

$(n-9)(n-2) = 0$

$n = 9, 2$

$\boxed{y_h = c_1 x^9 + c_2 x^2}$

$w_1 = x^9, \quad w_2 = x^2$

$\int x^{-15} = \frac{x^{-14}}{-14}$

$x^9 u'(x) + x^2 v'(x) = 0$

$9x^8 u'(x) + 2x v'(x) = \frac{4}{x^{14}(1)}$

$x^{-22} = \frac{x^{-21}}{-21}$

$$u'(x) = \frac{\begin{vmatrix} 0 & x^2 \\ \frac{4}{x^{14}} & 2x \end{vmatrix}}{\begin{vmatrix} x^9 & x^2 \\ 9x^8 & 2x \end{vmatrix}} = \frac{0 - \frac{x^2 \cdot 4}{x^{14}}}{2x^{10} - 9x^{10}} = \frac{\cancel{-}4x^{-12}}{\cancel{-}7x^{10}} = \frac{4}{7x^{22}}$$

$$u(x) = \int u'(x) = \int \frac{4}{7x^{22}} = \frac{4}{7}\frac{x^{-21}}{(-21)} = \frac{-4}{147}x^{-21}$$

$x^9\left(\frac{4}{7x^{22}}\right) + x^2 v'(x) = 0$

$x^2 v'(x) = -x^9\left(\frac{4}{7x^{22}}\right)$

$v'(x) = \frac{-x^7 \cdot 4}{7x^{22}} = -\frac{4}{7x^{15}}$

$v(x) = \int -\frac{4dx}{7x^{15}} = \frac{\cancel{-}4}{7}\frac{x^{-14}}{(\cancel{-}14)} = \frac{4x^{-14}}{7(14)} = \frac{2}{49}x^{-14}$

$y_p = w_1 u(x) + w_2 v(x) = -x^9\left(\frac{4}{147}x^{-21}\right) + x^2\left(\frac{2}{49}x^{-14}\right)$

$= \frac{-4}{147}x^{-12} + \frac{2}{49}x^{-12}$

$\boxed{y_p = \frac{2}{147}x^{-12}}$

$\therefore y_g = y_h + y_p$

$\boxed{y_g = c_1x^9 + c_2x^2 + \frac{2}{147}x^{-12}}$

Sneha

QUESTION 4. (10 points) Solve $y^{(2)} + \frac{-2}{2x+1}y' = \frac{-(2x+1)^2}{(x^2+x+1)^2}$

$W = y'$

$W' = y''$

$2x+1 = u$

$2dx = du$

$2\,dx = du$

$$W' + \frac{-2}{(2x+1)}W = \frac{-(2x+1)^2}{(x^2+x+1)^2}$$

$$b_0 = \frac{-2}{2x+1}$$

$$\int b_0 \cdot dx = -\int \frac{2}{2x+1}dx = -\int \frac{du}{u} = -\ln u = -\ln(2x+1)$$

$$e^{-\ln(2x+1)} = e^{\ln(2x+1)^{-1}} = \frac{1}{2x+1}$$

$$W = \frac{\int \frac{1}{\cancel{(2x+1)}}\left(\frac{-(2x+1)^{\cancel{2}}}{(x^2+x+1)^2}\right)dx}{\frac{1}{2x+1}}$$

$$u^{-2} = \frac{u^{-1}}{-1}$$

$$= \frac{-\int \frac{2x+1}{(x^2+x+1)^2}dx}{\frac{1}{2x+1}}$$

$(x^2+x+1) = u$

$(2x+1)\,dx = du$

✓

$$= \frac{-\int \frac{du}{u^2}}{\frac{1}{2x+1}} = \frac{-\left(\frac{u^{-1}}{-1}\right)+C}{\frac{1}{2x+1}} = \frac{(x^2+x+1)^{-1}+C}{1/2x+1}$$

$$\boxed{W = \frac{2x+1}{x^2+x+1} + (2x+1)C}$$

$y = \int W$

$= \int \frac{2x+1}{x^2+x+1}dx + \int (2x+1)C\;dx$

$= \int \frac{du}{u} + \left(\frac{2x^2}{2} + x\right)C$

$y = \ln|x^2+x+1| + (x^2+x)C + C_1$

Shena

Bernoulli

QUESTION 5. (10 points) Solve $y' + \frac{-1}{3}y = (\frac{-1}{3}cos(3x) + sin(3x))y^4$

$n=4$

$1-n = 1-4 = {}^{-}3$

$w = y^{-3}$

$w' + \frac{-1}{\cancel{3}}(-\cancel{3})w = -3(-\frac{1}{3}\cos(3x) + \sin(3x))$

$w' + w = \cos 3x - 3\sin(3x)$

$e^{\int 1 dx} = e^x$

$w = \frac{\int e^x(\cos 3x - 3\sin 3x)\,dx}{e^x} = \frac{e^x \cos 3x + c}{e^x}$

$w = \cos 3x + ce^{-x}$

$w = y^{-3}$

$\boxed{y = w^{-1/3} = (\cos 3x + ce^{-x})^{-1/3}}$

~~$w = y^{1-4} = y^{-3}$~~

~~$1 - n = -3$~~

~~$w' - \frac{1}{\cancel{3}}(-3)w = (-3)(-\frac{1}{3}\cos 3x + \sin 3x)$~~

~~$w' + w = \cos 3x - 3\sin 3x$~~

~~$w = \frac{\int e^x(\cos 3x - 3\sin 3x)}{e^x} = \frac{e^x \cos 3x + c}{e^x}$~~

~~$= \cos 3x + ce^{-x}$~~

~~$y = (\cos 3x + ce^{-x})^{-1/3}$~~

(-7,1) $y' \cdot 7y + y - 7y$

sneha

QUESTION 6. (10 points) Given Given $y' = y^3 - 6y^2 - 7y$. Find the critical points. Then classify each as stable or semi-stable or unstable. Roughly, sketch the solution to the DE if $y(0) = -0.5$

$y' = y(y^2 - 6y - 7)$

to find critical points

$y(y^2 - 6y - 7) = 0$

$y = 0, (y-7)(y+1) = 0$

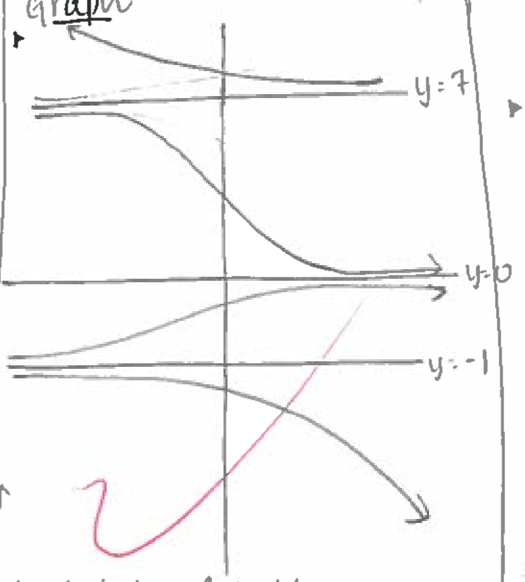
$y = 7, -1$

$- - - - + + + + - - - - - + + +$

$-1 \quad 0 \quad 7$

$y = -2$, $y' = -18$ | $y = -0.75$, $y' = +ve$ | $y = 1$

Graph



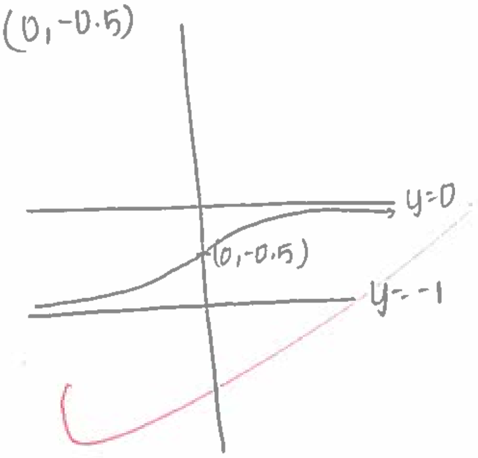
y = 7 is semistable.

y = 0 is stable

y = -1 is unstable

$y(0) = -0.5)$

$(0, -0.5)$



QUESTION 7. (10 points) Is $y' = \frac{-(y^2+4x^3+e^x+10)}{2yx+e^y+2y-10}$ exact? If yes, then solve it. If no, then find a method that will help us to solve it.

↗fx
↳fy

$dy(2yx + e^y + 2y - 10) + dx(y^2 + 4x^3 + e^x + 10) = 0$.

① $f_{xy} = 2y + 0 = 2y$

$F_{yx} = 2y + 0 = 2y$

$f_{xy} = f_{yx}$

$\therefore$ the equation is exact.

② choose f_y or f_x

$$\int f_y\,dy = \int 2yx + e^y + 2y - 10\,dy = \frac{\cancel{2}y^2}{\cancel{2}}x + e^y + \frac{\cancel{2}y^2}{\cancel{2}} - 10y + h(x)$$

$$F(x,y) = y^2x + e^y + y^2 - 10y + h(x)$$

③ $F_x = y^2 + h'(x)$

$\cancel{y^2} + 4x^3 + e^x + 10 = \cancel{y^2} + h'(x)$

$h'(x) = 4x^3 + e^x + 10$

$h(x) = \int 4x^3 + e^x + 10 = \frac{4x^4}{4} + e^x + 10x + C$

$= x^4 + e^x + 10x + C$

$$\boxed{y^2x + e^y + y^2 - 10y + x^4 + e^x + 10x + C = 0}$$

oriena

QUESTION 8. (10 points) An ice-cream cake with initial temperature 0C is placed in a room that has constant temperature 20C. If after 2 minutes, the temperature of the cake is 4C. a) How long will it take for the cake to reach the room temperature? b) What is the temperature of the cake after 30 minutes?

$\frac{dT}{dt} = k(T-Tc)$

$\int \frac{dT}{kT-kTc} = \int dt$

$\frac{1}{k}\ln|kT-kTc| = t+C$

$\ln|kT-kTc| = k(t+C)$

$kT-kTc = e^{k(t+C)}$

$T-Tc = \frac{e^{kt+C}}{k}$

$\boxed{T = \frac{e^{kt+C}}{k} + Tc}$

$T = e^{kt}(c_1) + Tc$

$T(0) = 0$

$0 = e^0(c_1) + 20$

$\boxed{c_1 = -20}$

$T(2) = 4$

$4 = e^{2k}(-20) + 20$

$\frac{-16}{-20} = e^{2k}$

$2k = \ln\left(\frac{16}{20}\right)$

~~2k = -0.~~

$\boxed{k = -0.112}$

$Tc = 20C$ | $T(2) = 4C$

$T(0) = 0$

a) $T = Tc$

$\cancel{Tc} = e^{-0.112t}(-20) + \cancel{Tc}$

$e^{-0.012t}(-20) = 0$

$e^{-0.112t} = 0$

this is not possible as e^a can never be 0. as a result there exist m time where $T = Tc$

b) $t = 30$

$T = e^{-0.112(30)}(-20) + 20$

$= 19.31C$

QUESTION 9. (15 points) Let $A(t)$ be the amount of salt at any time t. A 50-gal tank initially holds 10 gallons of fresh water (i.e. $A(0) = 0$). A mixture containing 1 kg of salt per gallon is poured into the tank at the rate 4 gal/min, while the well stirred mixture leaves the tank at rate 2 gal/min. a) Find A(t). b) Find the amount of salt at the moment of overflow? Find the concentration of salt per gallon after 10 minutes?

$\frac{dA}{dt} =$ rate of salt (in) − rate of salt (out)

$= 4 \times 1 - 2$ conc. of salt

conc. of salt $= \frac{A(t)}{10+4t-2t} = \frac{A(t)}{10+2t}$

$\frac{dA}{dt} = (4 \times 1) - \frac{2A(t)}{10+2t} = 4 - \frac{A(t)}{5+t}$

$A' + \frac{A(t)}{5+t} = 4$

$5+t = u$

$dt = du$

$\int \frac{du}{u} = \ln u$

b0 $= \frac{1}{5+t}$

$e^{\int 1/5+t\, dt} = e^{\ln|5+t|} = 5+t$

$A(t) = \frac{\int (5+t)\, 4\, dt}{(5+t)} = \frac{\int 20+4t\, dt}{5+t} = \frac{20t + \frac{\cancel{4}^{2}t^2}{\cancel{2}} + C}{5+t}$

$\boxed{A(t) = \frac{20t+2t^2+C}{5+t}}$

PTO →

Faculty information

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Excellent

$$\frac{100}{100}$$

Exam II, MTH 205, Fall 2014

Ayman Badawi

QUESTION 1. (10 points) Solve $\frac{dy}{dx} = \frac{1}{-x+2y+y^2}$

$$\frac{dx}{dy} = -x + 2y + y^2$$

$$x' + x = 2y + y^2 \quad (\text{1st order linear})$$

$$e^{\int b_0(y)dy} = e^{\int 1 dy} = e^y$$

$$x = \frac{\int e^y(\overset{f'(y)}{2y} + \overset{f(y)}{y^2})\,dy}{e^y} = \frac{e^y \cdot y^2 + c}{e^y}$$

$$\underline{x = y^2 + ce^{-y}}$$

✓

QUESTION 2. (10 points) Solve $\frac{dy}{dx} = \frac{-9x+3y+e^{(-3x+y)}}{-3x+y}$ $= \cancel{-\frac{(9x-3y-e^{(-3x+y)})}{-3x+y}}$ f_x

~~$f_x = 9x - 3y - e^{(-3x+y)}$~~ $f_{xy} = \cancel{-3} - (e^{-3x} e^y)$ (Not exact)

~~$f_y = -3x + y$, $f_{yx} = -3$~~

Let $w = -3x + y$

$$w' = -3 + y'$$

$$\frac{dw}{dx} = -3 + \frac{-9x + 3y + e^{(-3x+y)}}{-3x+y}$$

$$\frac{dw}{dx} = \frac{-3(-3x+y) - 9x + 3y + e^{(-3x+y)}}{-3x+y}$$

$$\frac{dw}{dx} = \frac{\cancel{9x} - \cancel{3y} - \cancel{9x} + \cancel{3y} + e^{(-3x+y)}}{-3x+y} = \frac{e^{(-3x+y)}}{-3x+y} = \frac{e^w}{w}$$

′	∫
w +	e^{-w}
1	$-e^{-w}$
0	e^{-w}

$$\int we^{-w}\,dw = \int dx \quad (\text{separable})$$

$$-(we^{-w} + e^{-w}) = x + c$$

✓

$$\underline{-\left((-3x+y)e^{-(-3x+y)} + e^{-(-3x+y)}\right) = x + c}$$

Manreet Kaur

QUESTION 3. (15 points) Solve $y^{(2)} - \frac{10}{x}y' + \frac{18}{x^2}y = \frac{4}{x^{14}}$ 2nd Order

$$y'' - \frac{10}{x}y' + \frac{18}{x^2}y = \frac{4}{x^{14}}$$

$y_g = y_h + y_p$, $\quad y_h = ?$

$$y'' - \frac{10}{x}y' + \frac{18}{x^2}y = 0 ,$$

Let $y = x^n$

$y' = nx^{n-1}$

$y^{(2)} = (n^2-n)x^{n-2}$

$$(n^2-n)x^{n-2} - \frac{10}{x}n(x^{n-1}) + \frac{18}{x^2}x^n = 0$$

$$x^{n-2}(n^2-n-10n+18) = 0$$

$$n^2 - 11n + 18 = 0$$

$$n^2 - 2n - 9n + 18 = 0$$

$$n(n-2) - 9(n-2) = 0$$

$$\underline{\underline{n = 9}} \quad \text{and} \quad \underline{\underline{n = 2}} ✓$$

S = -11

P = 18

-9 -2

$$y_h = c_1 \underline{x^9} + c_2 \underline{x^2}$$

$w_1 = x^9$

$w_2 = x^2$

Assume, $y_p = w_1(x)\,u(x) + w_2(x)\,v(x)$

$w_1 u'(x) + w_2 v' = 0$

$w_1' u'(x) + w_2' v' = \frac{k(x)}{a_2(x)}$

$$x^9 u'(x) + x^2 v'(x) = 0$$

$$9x^8 u'(x) + 2x\,v'(x) = \frac{4}{x^{14}}$$

$$u' = \frac{\begin{vmatrix} 0 & x^2 \\ \frac{4}{x^{14}} & 2x \end{vmatrix}}{\begin{vmatrix} x^9 & x^2 \\ 9x^8 & 2x \end{vmatrix}} = \frac{-\frac{4x^2}{x^{14}}}{2x^{10} - 9x^{10}} = \frac{-\frac{4}{x^{12}}}{-7x^{10}} = \frac{-4}{x^{12}} \times \frac{-1}{7x^{10}}$$

$$u'(x) = \underline{\underline{\frac{4}{7x^{22}}}} ✓$$

$$x^9\left(\frac{4}{7x^{22}}\right) + x^2 v'(x) = 0$$

$$\frac{4}{7x^{13}} = -x^2 v'(x)$$

$-\frac{4}{7x^{13} \times x^2}$

$-\frac{4}{7x^{13} x^2}$

$$\longleftarrow \quad \underline{\underline{v'(x) = -\frac{4}{7x^{15}}}} ✓$$

QUESTION 4. (10 points) Solve $y^{(2)} + \frac{-2}{2x+1}y' = \frac{-(2x+1)^2}{(x^2+x+1)^2}$

$a_0(x) = 0$
2nd order, case I.

Let $w = y'$

$\therefore y'' = w'$

$$w' + \underbrace{\frac{-2}{2x+1}}_{b_0(x)} w = \underbrace{-\frac{(2x+1)^2}{(x^2+x+1)^2}}_{f(x)}$$ 1st order linear

$$e^{\int b_0(x)\,dx} = e^{\int -\frac{2}{2x+1}dx} = e^{-1\int \frac{2}{2x+1}dx}$$

$$= e^{-1\ln|2x+1|}$$

$$= \underline{\underline{(2x+1)^{-1}}}$$

Let $2x+1 = t$
$2\,dx = dt$
$e^{-2\int \frac{dt}{t}}$
$= e^{-2\ln|t|}$

$$W = \frac{\int e^{\int b_0(x)dx} \cdot f(x)\,dx}{e^{\int b_0(x)dx}} = \frac{\int (2x+1)^{-1} \times \frac{-(2x+1)^2}{(x^2+x+1)^2}dx}{(2x+1)^{-1}}$$

$$\cancel{W = -\frac{\int \frac{1}{(x^2+x+1)^2}dx}{(2x+1)^{-2}}}$$

$$W = -\frac{\int \frac{2x+1}{(x^2+x+1)^2}dx}{(2x+1)^{-1}}$$

By subst

$x^2+x+1 = t$
$(2x+1)dx = dt$
$-\int \frac{dt}{t^2}$
$= -\left[-\frac{1}{t}\right]$
$= 1/t$
$= \frac{1}{x^2+x+1}$

$$W = \frac{\frac{1}{x^2+x+1} + c}{(2x+1)^{-1}}$$

$$w = \frac{2x+1}{(x^2+x+1)} + c\,(2x+1)$$

$$\therefore y = \int w\,dx = \int \left(\frac{2x+1}{x^2+x+1} + c(2x+1)\right)dx$$

$$= \int \frac{\overset{f'(x)}{2x+1}}{\underset{f(x)}{x^2+x+1}}dx + c\int (2x+1)\,dx$$

(By subst)

$$y = \ln|x^2+x+1| + c(x^2+x) + c_1$$

Let $x^2+x+1 = t$
$(2x+1)dx = dt$
$\int \frac{dt}{t} = \ln|t|$
$= \ln|x^2+x+1|$

$a_1(x) w' + (a_0(x)/(1-n)) w = k(x)/(1-n)$

Maneet

QUESTION 5. (10 points) Solve $y' + \frac{-1}{3}y = (\frac{-1}{3}cos(3x) + sin(3x))y^4$

1st order non-linear (Bernoulli's)

$f(x)$

$n = 4, \quad 1-n = 1-4 = \underline{\underline{-3}}.$

$\therefore w = y^{-3}$

$w' + \frac{-1}{3}(-3)w = \left(-\frac{1}{3}\cos(3x) + \sin(3x)\right)(-3)$

$\underbrace{w' + w}_{b_0(x)} = \underbrace{\cos 3x + (-3)\sin 3x}_{f(x)}$ 1st order linear

$e^{\int b_0(x)dx} = e^{\int 1 dx} = e^x$

$w = \dfrac{\int e^x . (\overset{f(x)}{\cos 3x} \overset{f'(x)}{- 3\sin 3x})dx}{e^x}$

Let $\cos 3x = t$

$-3\sin 3x\, dx = dt$

$w = \dfrac{e^x \cos 3x + c}{e^x} = \underline{\cos 3x + ce^{-x}}$

$\therefore y^{-3} = w, \quad y = w^{-1/3}$

$y = \underline{(\cos 3x + ce^{-x})^{-1/3}}$

Mameet

QUESTION 6. (10 points) Given Given $y' = y^3 - 6y^2 - 7y$. Find the critical points. Then classify each as stable or semi-stable or unstable. Roughly, sketch the solution to the DE if $y(0) = -0.5$

$y^3 - 6y^2 - 7y = 0$

$y(y^2 - 6y - 7) = 0$

$y = 0$, $y^2 - 6y - 7 = 0$

$y^2 + y - 7y - 7 = 0$

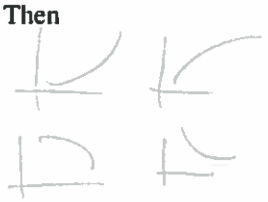
$y(y+1) - 7(y+1) = 0$

$y = 7$ and $y = -1$

$S = -6$

$P = -7$

$-7 \quad 1$



Critical points are $y = 0, y = 7, y = -1$

$-\infty$ —— -1 —— 0 —— 7 —— $+\infty$

−ve, +ve, −ve, +ve

$(-\infty, -1)$

$y' = (-2)^3 - 6(-2)^2 - 7(-2)$

$= -ve$

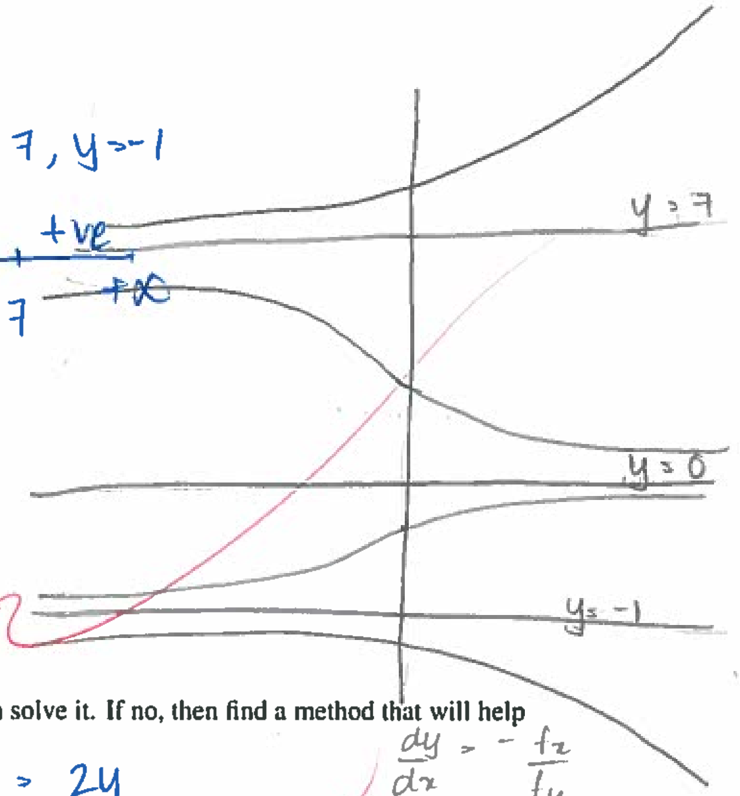
$(-1, 0)$, $y' = (-0.5)^3 - 6(-0.5)^2 - 7 \times (-0.5)$

$= +ve$.

$(0, 7)$, $y' = (1)^3 - 6(1)^2 - 7(1) = -ve$

$(7, \infty)$, $y' = (9)^3 - 6(9)^2 - 7(9) = +ve$

P.T.O



QUESTION 7. (10 points) Is $y' = \frac{-(y^2+4x^3+e^x+10)}{2yx+e^y+2y-10}$ exact? If yes, then solve it. If no, then find a method that will help us to solve it.

$\frac{dy}{dx} = -\frac{f_x}{f_y}$

① $f_x = y^2 + 4x^3 + e^x + 10$, $f_{xy} = 2y$

$f_y = 2xy + e^y + 2y - 10$, $f_{yx} = 2y$

$f_{xy} = f_{yx}$, ∴ It is exact

② $\int f_x \, dx = \int (y^2 + 4x^3 + e^x + 10) \, dx$

$f(x,y) = y^2 x + \frac{4x^4}{4} + e^x + 10x + h(y)$

$f(x,y) = xy^2 + x^4 + e^x + 10x + h(y)$

$f(x,y)_y = 2xy + h'(y)$

$\cancel{2xy} + e^y + 2y - 10 = \cancel{2xy} + h'(y)$

$h'(y) = e^y + 2y - 10$

$h(y) = \int h'(y) dy = \int (e^y + 2y - 10) dy$

$= e^y + y^2 - 10y + c$

∴ $\cancel{f(x,y) =} xy^2 + x^4 + e^x + 10x + e^y + y^2 - 10y + \cancel{c} = C$

Maneet

QUESTION 8. (10 points) An ice-cream cake with initial temperature 0C is placed in a room that has constant temperature 20C. If after 2 minutes, the temperature of the cake is 4C. a) How long will it take for the cake to reach the room temperature? b) What is the temperature of the cake after 30 minutes?

$T(0) = 0$ $T_c = 20C$

T = ice cream temp. , T_c = room temp.

$\frac{dT}{dt} = K(T-T_c)$, $\int \frac{dT}{K(T-T_c)} = \int dt$, $\frac{1}{K} \ln(T-T_c) = t+c$

$\ln(T-T_c) = K(t+c)$

$T = e^{K(t+c)} + T_c$

$T = e^{Kt} c_1 + T_c$

$T = e^{Kt} c_1 + T_c$

$T(0) = 0$, given, $T_c = 20$

$0 = e^0 c_1 + 20$

$0 = c_1 + 20$, $\underline{\underline{c_1 = -20}}$

Also, $T(2) = 4$, $T = e^{Kt}(-20) + 20$

$4 = e^{K(2)}(-20) + 20$

$-16 = e^{2K}(-20)$

$\frac{\ln\left(\frac{16}{20}\right)}{2} = K = \underline{\underline{-0.1116}}$

$\leftarrow$ P.T.O

QUESTION 9. (15 points) Let $A(t)$ be the amount of salt at any time t. A 50-gal tank initially holds 10 $\overset{V_0}{\text{gallons}}$ of fresh water (i.e. $A(0) = 0$). A mixture containing 1 kg of salt per gallon is poured into the tank at the rate 4 gal/min, while the well stirred mixture leaves the tank at rate 2 gal/min. a) Find A(t). b) Find the amount of salt at the moment of overflow? Find the concentration of salt per gallon after 10 minutes?

$\frac{dA}{dt} = \text{rate (in)} - \text{rate (out)}$

$= (4 \times 1) - (2 \times \text{conc})$

$\text{conc} = \frac{A(t)}{10+4t-2t} = \frac{A(t)}{10+2t}$

$\frac{dA}{dt} = 4 - \frac{2A(t)}{10+2t}$, $A' + \frac{2A}{10+2t} = 4$ 1st order linear

$A' + \frac{A}{5+t} = 4$

$e^{\int \frac{1}{5+t} dt} = e^{\ln|5+t|} = \underline{\underline{5+t}}$

$A = \frac{\int (5+t) \cdot 4 \, dt}{5+t}$

$= \frac{4\left(5t + \frac{t^2}{2}\right) + c}{5+t}$

$A = \frac{20t + 2t^2 + c}{5+t}$

But $A(0) = 0$;

$\therefore 0 = \frac{20(0) + 2(0)^2 + c}{5+0}$, $\underline{\underline{0 = c}}$

$\therefore A = (20t + 2t^2)/(5+t)$

$\leftarrow$ P.T.O

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Name Yasmine Abbas, ID 51424

Exam II, MTH 205, Fall 2014

Ayman Badawi

Excellent

$$\frac{100}{100}$$

QUESTION 1. (10 points) Solve $\dfrac{dy}{dx} = \dfrac{1}{-x+2y+y^2}$

$$\frac{dx}{dy} = -x + 2y + y^2$$

$$x' + x = y^2 + 2y$$

$$x = \frac{\int e^{\int dy}(y^2+2y)\ dy}{e^{\int dy}} = \frac{\int e^{y}(y^2+2y)\,dy}{e^{y}}$$

$$\Rightarrow x = \frac{y^2e^y + c}{e^y}$$

$$\boxed{x = y^2 + ce^{-y}}$$

QUESTION 2. (10 points) Solve $\dfrac{dy}{dx} = \dfrac{-9x+3y+e^{(-3x+y)}}{-3x+y}$ =

$$\frac{dy}{dx} = \frac{3(-3x+y) + e^{-3x+y}}{-3x+y}$$

$$\boxed{w = -3x+y}$$

$$\frac{dw}{dx} = -3 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{dw}{dx} + 3$$

$$\frac{dw}{dx} + 3 = \frac{3w + e^{w}}{w} = 3 + \frac{e^{w}}{w}$$

$$\frac{dw}{dx} = \frac{e^{w}}{w}$$

$$\int \frac{w\,dw}{e^{w}} = \int dx$$

$$\int we^{-w}\,dw = x + C$$

$$= -we^{-w} - e^{-w} = x + C$$

$$\boxed{-(-3x+y)\,e^{-(-3x+y)} - e^{-(-3x+y)} = x + C}$$

	$\int$
w	e^{-w}
1	$-e^{-w}$
0	e^{-w}

Yassmine Abbas

QUESTION 3. (15 points) Solve $y^{(2)} - \frac{10}{x}y' + \frac{18}{x^2}y = \frac{4}{x^{14}}$

x^{-1} x^{-2}

$y = x^n$

$y' = nx^{n-1}$

$y'' = (n^2 - n)x^{n-2}$

$(n^2 - n) - 10n + 18 = 0$

$n^2 - 11n + 18 = 0$

$(n-9)(n-2) = 0$

$n = 9 , n = 2$

$\boxed{y_h = c_1x^9 + c_2x^2}$

$y_1 = x^9 \quad , \quad y_1' = 9x^8$

$y_2 = x^2 \quad , \quad y_2' = 2x$

$x^9 u'(x) + x^2 v'(x) = 0 \implies v'(x) = -\frac{x^9 u'(x)}{x^2} = -x^7 u'(x)$

$9x^8 u'(x) + 2x\, v'(x) = \frac{4}{x^{14}}$

$$u'(x) = \frac{\begin{vmatrix} 0 & x^2 \\ \frac{4}{x^{14}} & 2x \end{vmatrix}}{\begin{vmatrix} x^9 & x^2 \\ 9x^8 & 2x \end{vmatrix}} = \frac{-x^2\left(\frac{4}{x^{14}}\right)}{2x^{10} - 9x^{10}} = \frac{-\frac{4}{x^{12}}}{-7x^{10}} = \frac{4}{7}\frac{1}{x^{22}}$$

$u(x) = \frac{4}{7}\int x^{-22}dx$

$= \frac{4}{7}\frac{x^{-21}}{-21} = \boxed{-\frac{4}{147}\frac{1}{x^{21}}}$

$\implies v'(x) = -x^7 \cdot \frac{4}{7}\frac{1}{x^{22}} = \frac{-4}{7}\frac{1}{x^{15}}$

$v(x) = -\frac{4}{7}\int x^{-15}dx = -\frac{4}{7}\frac{x^{-14}}{-14}$

$= \boxed{\frac{2}{49}\frac{1}{x^{14}}}$

$\implies y_g = y_h + y_p = c_1x^9 + c_2x^2 - \frac{4}{147}x^{-21} \cdot x^9 + \frac{2}{49}x^{-14} \cdot x^2$

$= c_1x^9 + c_2x^2 - \frac{4}{147}x^{-12} + \frac{2}{49}x^{-12}$

$y = c_1x^9 + c_2x^2 + \frac{2}{} x^{-12}$

QUESTION 4. (10 points) Solve $y^{(2)} + \frac{-2}{2x+1}y' = \frac{-(2x+1)^2}{(x^2+x+1)^2}$

$$w = y'$$

$$\frac{dw}{dx} = y''$$

$$w' - \frac{2}{2x+1}\, w = \frac{-(2x+1)^2}{(x^2+x+1)^2}$$

$$e^{\int \frac{-2}{2x+1}\, dx} = e^{-\ln|2x+1|} = \frac{1}{2x+1}$$

$$w = \frac{-\int \frac{(2x+1)^2}{(x^2+x+1)^2} \cdot \frac{1}{2x+1}\, dx}{1/2x+1} = \frac{-\int \frac{(2x+1)}{(x^2+x+1)^2}\, dx}{1/2x+1}$$

$$u = x^2+x+1$$

$$du = 2x+1$$

$$w = \frac{-\int \frac{du}{u^2}}{1/2x+1} = \frac{\frac{1}{u} + C}{1/2x+1} = \frac{\frac{1}{x^2+x+1} + C}{1/2x+1}$$

$$\Rightarrow w = \frac{2x+1}{x^2+x+1} + C(2x+1)$$

$$y = \int w\, dw = \int\left(\frac{2x+1}{x^2+x+1} + C(2x+1)\right) dx$$

$$\boxed{y = \ln|x^2+x+1| + C(x^2+x) + C_1}$$

✓

Yassmine.

QUESTION 5. **(10 points)** Solve $y' + \frac{-1}{3}y = (\frac{-1}{3}cos(3x) + sin(3x))y^4$

$n = 4$

$1 - n = -3$

$w = y^{-3} \rightarrow y = \sqrt[3]{1/w}$

$$w' + (-3)\left(\frac{-1}{3}\right)w = -3\left(-\frac{1}{3}\cos(3x) + \sin(3x)\right)$$

$$w' + w = \cos(3x) - 3\sin(3x)$$

$$w = \frac{\int e^x (\cos(3x) - 3\sin(3x))\,dx}{e^x}$$

$$w = \frac{e^x \cos(3x) + C}{e^x}$$

$$\Rightarrow w = \cos(3x) + Ce^{-x}$$

$$\boxed{y = \sqrt[3]{\frac{1}{\cos(3x) + Ce^{-x}}}}$$

QUESTION 6. (10 points) Given Given $y' = y^3 - 6y^2 - 7y$. Find the critical points. Then classify each as stable or semi-stable or unstable. Roughly, sketch the solution to the DE if $y(0) = -0.5$

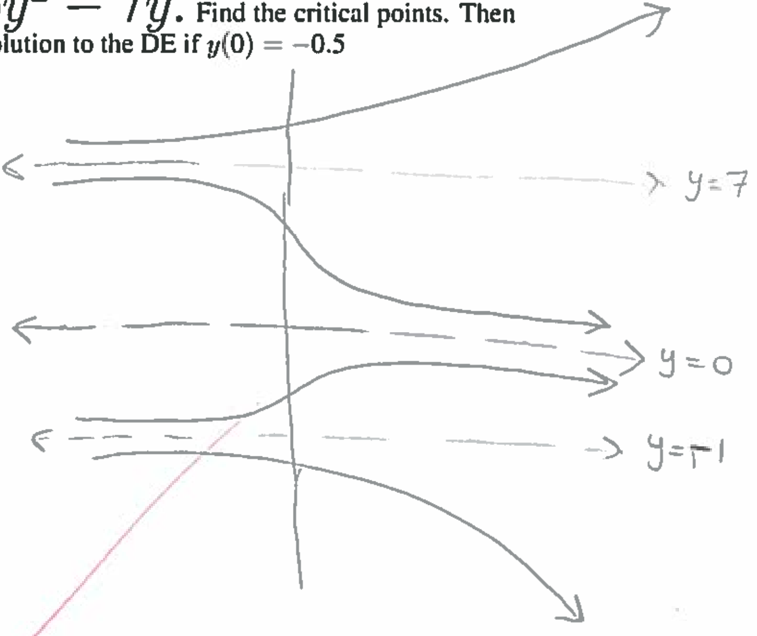
critical points : $y' = 0$

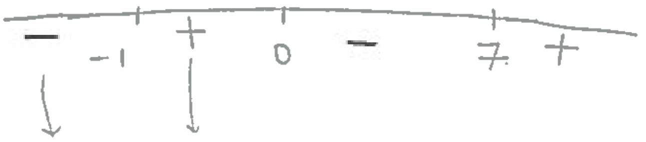
$y^3 - 6y^2 - 7y = 0$

$y(y^2 - 6y - 7) = 0$

$\underline{y = 0}$ $\quad (y-7)(y+1) = 0$

$\underline{y = 7} \quad \underline{y = -1}$





-0.5

$- 2(4 + 12 - 7)$

(a)

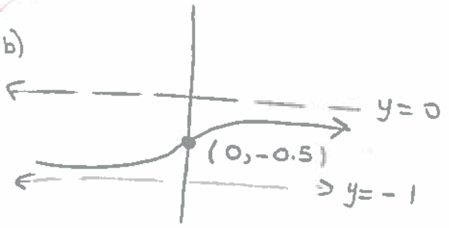
$\Rightarrow y = -1 \Rightarrow$ unstable

$y = 0 \Rightarrow$ stable

$y = 7 \Rightarrow$ unstable

(b)

$y(0) = -0.5$ →



QUESTION 7. (10 points) Is $y' = \frac{-(y^2+4x^3+e^x+10)}{2yx+e^y+2y-10}$ exact? If yes, then solve it. If no, then find a method that will help us to solve it.

$$\frac{dy}{dx} = -\frac{(y^2+4x^3+e^x+10)}{(2yx+e^y+2y-10)} = -\frac{fx}{fy}$$

$(fx)_y = 2y$

$f(y)_x = 2y$ $\quad \Rightarrow (fx)y = (fy)x \Rightarrow$ yes! exact/Implicit)

$$(2yx + e^y + 2y - 10)\,dy + (y^2 + 4x^3 + e^x + 10)\,dx = 0$$

$$f(x,y) = \int fy\,dy = \int (2yx + e^y + 2y - 10)\,dy = (y^2x + e^y + y^2 - 10y + h(x))$$

$$fx = y^2 + h'(x)$$

$\cancel{y^2} + 4x^3 + e^x + 10 = \cancel{y^2} + h'(x) \Rightarrow h'(x) = 4x^3 + e^x + 10$

$h(x) = \int (4x^3 + e^x + 10)\,dx = x^4 + e^x + 10x + C$

$\Rightarrow \boxed{~~f(x,y) =~~ y^2x + e^y + y^2 - 10y + x^4 + e^x + 10x + C = 0}$

Yossmine

QUESTION 8. (10 points) An ice-cream cake with initial temperature 0C is placed in a room that has constant temperature 20C. If after 2 minutes, the temperature of the cake is 4C. a) How long will it take for the cake to reach the room temperature? b) What is the temperature of the cake after 30 minutes?

$T_c = 20°C$

$T(0) = 0°C$

$T(2) = 4°C$

$t = ?\quad T = 20°C$

$T = ?\quad t = 30min$

$$\frac{dT}{dt} = K(T - T_c)$$

$$\frac{dT}{dt} = K(T - 20)$$

$$\ln|T - 20| = Kt + C$$

$$T - 20 = ae^{Kt}$$

$$\boxed{T = ae^{kt} + 20}$$

$$0 = ae^{0} + 20 \Rightarrow a = -20$$

$$4 = -20e^{2K} + 20 \Rightarrow e^{2K} = \frac{4-20}{-20} = \frac{4}{5}$$

$$\Rightarrow K = \ln(4/5)/2 = -0.112$$

$$\boxed{T = -20e^{-0.112t} + 20}$$

a) $20 = -20e^{-0.112t} + 20$

$e^{-0.112t} = 0 \Rightarrow$ never $\Rightarrow$ It will never reach 20°C

b) $T = -20e^{-0.112(30)} + 20$

$= \boxed{19.31°C}$

QUESTION 9. (15 points) Let $A(t)$ be the amount of salt at any time t. A 50-gal tank initially holds 10 gallons of fresh water (i.e. $A(0) = 0$). A mixture containing 1 kg of salt per gallon is poured into the tank at the rate 4 gal/min, while the well stirred mixture leaves the tank at rate 2 gal/min. a) Find A(t). b) Find the amount of salt at the moment of overflow? Find the concentration of salt per gallon after 10 minutes?

$A(0) = 0$

$V_0 = 10\ gal$

$$\frac{dA}{dt} = (1 \times 4) - (c \times 2)\ ,\quad c = \frac{A(t)}{10 + 4t - 2t} = \frac{A(t)}{10+2t}$$

$$\frac{dA}{dt} = 4 - \frac{2A(t)}{10+2t}$$

$$A' + \frac{A}{5+t} = 4$$

$$A(t) = \frac{\int e^{\int \frac{dt}{5+t}}(4)\,dt}{e^{\int \frac{dt}{5+t}}} = 4\frac{\int \overset{t+5}{e^{\ln|5+t|}}\,dt}{e^{\ln|5+t|}} = \frac{4\int(t+5)\,dt}{t+5} = \frac{4\left(\frac{t^2}{2} + 5t\right) +}{t+5}$$

$$A(0) = 0 \Rightarrow 0 = \frac{C}{5} \Rightarrow C = 0$$

a) $\boxed{A(t) = \frac{2t^2 + 20t}{t+5}}$

b) $V = 50 \Rightarrow$ over flow $\quad 50 = 10 + 2t \Rightarrow t = 20 \quad \Rightarrow A(20) = 48$ kg of salt

c) concentration $= \frac{A(10)}{10 + 2(10)} = \frac{80/3}{30} = 0.889$ kg salt/gallon

$\left(\frac{8}{9}\ kg/gal\right)$

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Name Nour AlJafari, ID 55471

Exam II, MTH 205, Fall 2014

Ayman Badawi

100
Excellent!

QUESTION 1. (10 points) Solve $\frac{dy}{dx} = \frac{1}{-x+2y+y^2}$

$$\frac{dx}{dy} = -x + 2y + y^2$$

$$x' + x = 2y + y^2$$

$$e^{\int dy} = e^y \qquad x = \frac{\int e^y (y^2 + 2y)}{e^y} = \frac{y^2 e^y + c}{e^y}$$

$$x = y^2 + c e^{-y}$$

QUESTION 2. (10 points) Solve $\frac{dy}{dx} = \frac{-9x+3y+e^{(-3x+y)}}{-3x+y}$

$$\frac{dy}{dx} = \frac{3\cancel{(-3x+y)}}{\cancel{-3x+y}} + \frac{e^{-3x+y}}{-3x+y}$$

$$\frac{dy}{dx} = 3 + \frac{e^{-3x+y}}{3x+y}$$

$$W = -3x+y$$

$$\frac{dw}{dx} = -3 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{dw}{dx} + 3$$

$$\frac{dw}{dx} + \cancel{3} = \cancel{3} + \frac{e^w}{w}$$

$$\int \frac{w\,dw}{e^w} = \int dx$$

$$w e^w - e^w = x + c$$

u	dv
w	+ e^w
1	e^w
0	e^w

$$\boxed{(-3x+y)e^{(-3x+y)} - e^{(-3x+y)} = x + c}$$

Nour.

QUESTION 3. (15 points) Solve $y^{(2)} - \frac{10}{x}y' + \frac{18}{x^2}y = \frac{4}{x^{14}}$

$y = x^n \quad y' = nx^{n-1}$

$y'' = (n^2-n)x^{n-2}$

$$(n^2-n)x^{n-2} - 10n\,x^{n-2} + 18x^{n-2} = 0$$

$$x^{(n-2)}(n^2 - n - 10n + 18) = 0$$

$$x^{n-2}(n^2 - 11n + 18) = 0$$

$$x^{n-2}(n-9)(n-2) = 0$$

$n = 9$

$n = 2$

$y_1 = x^9.$

$y_2 = x^2$

$y_h = c_1x^9 + c_2x^2$

$w_1(x) = x^9. \quad w_2(x) = x^2$

$w_1'(x) = 9x^8 \quad w_2'(x) = 2x$

$\frac{k(x)}{a_2(x)} = \frac{4}{x^{14}} = f(x)$

variation.

$$W_1(x)U'(x) + W_2(x)V'(x) = 0$$

① — $x^9U'(x) + x^2V'(x) = 0$

② — $\left(9x^8U'(x) + 2xV'(x) = \frac{4}{x^{14}}\right)\frac{1}{2}x$

$$x^9u'(x) + x^2v'(x) = 0$$

$$\frac{9}{2}x^9U'(x) + x^2V'(x) = \frac{2}{x^{13}}$$

$$\frac{-7}{2}x^9U(x) = \frac{-2}{x^{13}}$$

$$U'(x) = \frac{4}{7x^{22}}$$

$$x^9\frac{4}{7x^{22}} + x^2v'(x) = 0$$

$$x^2v'(x) = -\frac{4}{7x^{13}}$$

$$v'(x) = \frac{-4}{7x^{15}}$$

$$U(x) = \frac{4}{7}\int x^{-22}\,dx = \frac{-4x^{-21}}{147} + C.$$

$$V(x) = \frac{-4}{7}\int x^{-15}\,dx = \frac{2x^{-14}}{49} + c$$

$$y_p = W_1(x)U(x) + W_2(x)V(x)$$

$$= \frac{-4}{147}x^{-12} + \frac{2}{49}x^{-12}$$

$$Y_g = c_1x^9 + c_2x^2 - \frac{4}{147}x^{-12} + \frac{2}{49}x^{-12} = \boxed{c_1x^9 + c_2x^2 + \frac{2}{147}x^{-12}}$$

QUESTION 4. (10 points) Solve $y^{(2)} + \frac{-2}{2x+1}y' = \frac{-(2x+1)^2}{(x^2+x+1)^2}$

$w = y'$ $\qquad w' + \frac{-2}{2x+1} w = \frac{-(2x+1)^2}{(x^2+x+1)^2}$

$e^{\int \frac{-2}{2x+1} dx} = e^{-\ln 2x+1} = e^{\ln(2x+1)^{-1}} = \frac{1}{2x+1}$

$w = \dfrac{\int \frac{1}{\cancel{2x+1}} \cdot \frac{-(2x+1)^{\cancel{2}}}{(x^2+x+1)^2}}{1/2x+1} \rightarrow -\int \frac{2x+1}{(x^2+x+1)^2} dx$;

$u = x^2+x+1$

$du = 2x+1$

$= -\int \frac{du}{u^2} = \int -u^{-2} du$

$= \frac{-u^{-1}}{-1} = u^{-1} + c$

$\frac{1}{u} = \frac{1}{x^2+x+1} + c$

$w = \left(\frac{1}{x^2+x+1} + c\right)(2x+1)$

$w = \frac{2x+1}{x^2+x+1} + (2x+1)c$

$w = y'$

$y = \int w \, dx = \int \frac{2x+1}{x^2+x+1} + (2x+1)c \, dx$

$$\boxed{y = \ln(x^2+x+1) + (x^2+x)c + c_1}$$

✓

QUESTION 5. (10 points) Solve $y' + \frac{-1}{3}y = (\frac{-1}{3}cos(3x) + sin(3x))y^4$

$W = y^{1-n} = y^{1-4} = y^{-3}$ $\quad 1-n = -3$

$W' + W' = \cos 3x - 3\sin 3x$

$e^{\int dx} = e^x \quad W = \frac{\int e^x(\cos 3x - 3\sin 3x)}{e^x} = \frac{e^x \cos 3x + c}{e^x}$

$W = \cos 3x + Ce^{-x}$

$\frac{1}{y^3} = \cos 3x + Ce^{-x}$

$y^3 = \frac{1}{\cos 3x + Ce^{-x}}$

$y = \sqrt[3]{\frac{1}{\cos 3x + ce^{-x}}}$

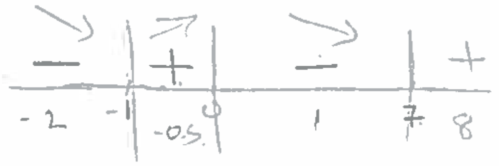
QUESTION 6. (10 points) Given Given $y' = y^3 - 6y^2 - 7y$. Find the critical points. Then classify each as stable or semi-stable or unstable. Roughly, sketch the solution to the DE if $y(0) = -0.5$

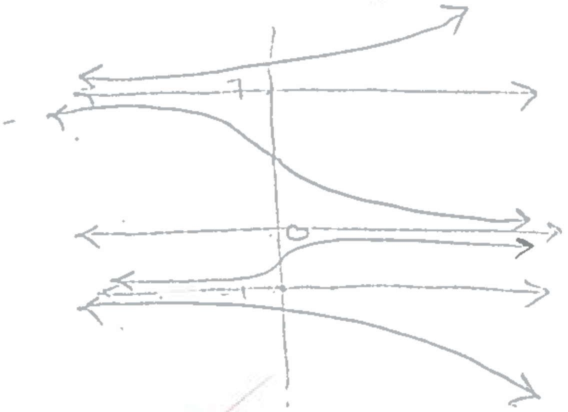
$0 = y^3 - 6y^2 - 7y$

$0 = \cancel{y}(y^2 - 6y - 7)$

$0. = y(y-7)(y+1)$

CP → $y = 0,\ y = 7,\ y = -1$



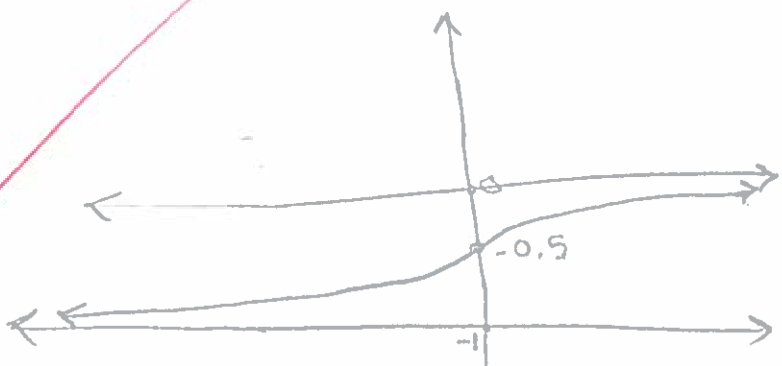


$y = -1$ → unstable

$y = 0$ → stable

$y = 7$ → unstable

$y(0) = -0.5$



QUESTION 7. (10 points) Is $y' = \frac{-(y^2+4x^3+e^x+10)}{2yx+e^y+2y-10}$ exact? If yes, then solve it. If no, then find a method that will help us to solve it.

f_y f_x ✓

$dy(2yx + e^y + 2y - 10) + dx(y^2 + 4x^3 + e^x + 10) = 0$

$f_{yx} = 2y$ since $f_{yx} = f_{xy} \rightarrow 2y = 2y$ then it is exac

$f_{xy} = 2y$ $f(x,y) = \int f_y\,dy = \int 2yx + e^y + 2y - 10\,dy = xy^2 + e^y + y^2 - 10$
$+ h(x)$

$\cancel{y^2} + h'(x) = \cancel{y^2} + 4x^3 + e^x + 10$

$h'(x) = 4x^3 + e^x + 10$

$\int h'(x) = \int 4x^3 + e^x + 10\,dx = x^4 + e^x + 10x + c = h(x)$

$\boxed{f~~(x,y)~~\ xy^2 + e^y + y^2 - 10y + x^4 + e^x + 10x\ ~~=~~} = C$

Nour

$T(0) = 0 \quad T_c = 20 \quad T(2) = 4.$

QUESTION 8. (10 points) An ice-cream cake with initial temperature 0C is placed in a room that has constant temperature 20C. If after 2 minutes, the temperature of the cake is 4C. a) How long will it take for the cake to reach the room temperature? b) What is the temperature of the cake after 30 minutes?

$$\frac{dT}{dt} = k(T - T_c)$$

$$\frac{dT}{dt} = k(T - 20)$$

$$\int \frac{dT}{T-20} = \int k\,dt$$

$$\ln T - 20 = kt + c$$

$$T - 20 = e^{kt} \cdot e^{c}$$

$$T(t) = ce^{kt} + 20$$

$$T(0) = c\,e^{\overset{1}{\cancel{0}}} + 20 = 0$$

$$c = -20.$$

$$T(2) = 4 = -20e^{k(2)} + 20$$

$$-16 = -20e^{2k} \rightarrow \ln \frac{4}{5} = \ln e^{2k}$$

$$2k = -0.22314$$

$$k = -0.1116$$

$$T(t) = 20 - 20e^{-0.1116t}$$

ⓐ $T(t) = 20 = 20 - 20e^{-0.1116t}$

$\frac{1}{e^{0.1116t}} = 0$ at $t \rightarrow \infty$, it will never reach room temp.

ⓑ. $T(30) = 20 - 20e^{-0.1116(30)}$

$= 19.3$ C.

QUESTION 9. (15 points) Let $A(t)$ be the amount of salt at any time t. A 50-gal tank initially holds 10 gallons of fresh water (i.e. $A(0) = 0$). A mixture containing 1 kg of salt per gallon is poured into the tank at the rate 4 gal/min, while the well stirred mixture leaves the tank at rate 2 gal/min. a) Find A(t). b) Find the amount of salt at the moment of overflow? Find the concentration of salt per gallon after 10 minutes?

$$\frac{dA}{dt} = \text{Rate in} - \text{Rate out}$$

$$\frac{dA}{dt} = 4 - \frac{2A(t)}{10+2t}$$

$$A'(t) + \frac{2}{10+2t} A(t) = 4$$

$$e^{\int \frac{2}{10+2t} dt} = e^{\ln 10+2t} = 10+2t$$

$$\frac{\int 40 + 8t\,dt}{10+2t} = \frac{40t + 4t^2 + c}{10+2t} = A(t)$$

$$0 = \frac{c}{10} \Rightarrow c = 0.$$

a) $A(t) = \dfrac{40t + 4t^2}{10+2t}$

Concentration $= \dfrac{A(t)}{10 + 4t - 2t}$

$= \dfrac{A(t)}{10+2t}$

ⓒ $A(10) = \dfrac{40(10) + 4(10)^2}{10 + 2(10)}$

$= 26.7$ kg salt

Concentration $= \dfrac{26.7}{10 + 2(10)}$

$= 0.89$ kg/gallon salt

b) Moment of over flow

$V = 10 + 2t = 50$

$t = 20$ min

$$A(20) = \frac{40(20) + 4(20)^2}{10 + 2(20)}$$

$= 48$ kg salt

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